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NOTE

THE COMPLEXITY OF A CLASS OF INFINITE GRAPHS

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If $\mathcal{G}_k$ is the family of countable graphs with no k vertex (or edge) disjoint circuits ($1<k<\omega$) then there is a countable $\mathcal{G}'_k \leq \mathcal{G}_k$ such that every member of $\mathcal{G}_k$ is an (induced) subgraph of some member of $\mathcal{G}'_k$, but no finite $\mathcal{G}'_k$ suffices.

Given a class of graphs $\mathcal{G}$, we say that $\mathcal{G}$ has a *universal* element $G_0 \in \mathcal{G}$ if any other graph $G \in \mathcal{G}$ is isomorphic to a (not necessarily induced) subgraph of G_0. The theory of universal graphs was initiated by R. Rado who showed ([7], [8]) that there is a universal countable graph. Since then, there have been several results on the existence of universal graphs.

In all interesting examples $\mathcal{G}$ is *closed under containment*, i.e., $G \in \mathcal{G}$ implies that $G' \in \mathcal{G}$ for any $G' \subseteq G$. This condition is satisfied e.g. for all classes of graphs which can be obtained in the following way. Given a cardinal κ and a family $\mathcal{H}$ of so-called *forbidden subgraphs*, let $\mathcal{G}_\kappa(\mathcal{H})$ be defined as the class of all graphs with at most κ vertices containing no subgraph isomorphic to any element of $\mathcal{H}$.

It is well-known that there is a universal element in $\mathcal{G}_\omega(K_n)$ for $n=2,3,\ldots$, but not in $\mathcal{G}_\omega(K_\omega)$ (here K_n denotes the complete graph on n vertices). For extensions to larger cardinals see [6]. A. Hajnal and J. Pach proved that there is no universal graph in $\mathcal{G}_\omega(C_4)$ ([2]) where C_n denotes the circle on n vertices. This was extended to C_n ($n \geq 3$) by G. Cherlin and P. Komjáth ([1]), and to complete bipartite graphs by yours faithfully ([4]).

In [3] the existence of a universal element in $\mathcal{G}_\omega(\{C_n, C_{n+1}, \ldots\})$ and in $\mathcal{G}_\omega(P_n)$ was established (P_n = path on n vertices). The proof showed that the class in question splits into finitely many subclasses,i.e., there can be finitely many different types of structure, and all those subclasses have a natural universal element. The disjoint union of those universal elements will, of course, be universal for the whole class. The last argument is based on the fact that in those cases the forbidden subgraphs are connected. In [5] we started to investigate the problem arising when we drop the condition that the forbidden subgraphs be connected. For that reason we extended the definition of universality as follows. Let $c(\mathcal{G})$, the *complexity* of a class of graphs $\mathcal{G}$, be defined as the least cardinality of a subset $\mathcal{G}_0 \subseteq \mathcal{G}$ with the property that any element of $\mathcal{G}$ is isomorphic to a subgraph of some $G_0 \in \mathcal{G}_0$. Obviously, $\mathcal{G}$ has a universal element if and only if $c(\mathcal{G})=1$.

Let $\mathcal{G}_k$ denote the class of all countable graphs containing no k vertex-disjoint cycles. That is, using the above notation, $\mathcal{G}_k = \mathcal{G}_\omega(\mathcal{H}_k)$, where $\mathcal{H}_k$ stands for the

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family of all (finite) graphs consisting of k vertex-disjoint cycles. In particular, $\mathscr{G}_1$ is the class of all countable forests and $c(\mathscr{G}_1)=1$. Since $\mathscr{G}_k$ has continuum many elements, its complexity is at most 2^ω.

Theorem 1. *Let $1<k<\omega$, and let $\mathscr{G}_k$ be the class of all countable graphs containing no k vertex-disjoint cycles. Then $c(\mathscr{G}_k)=\omega$.*

Proof. First we show that $c(\mathscr{G}_k)\le\omega$.

Let G be a fixed countable graph without k vertex-disjoint cycles. Fix a finite subset $K\subseteq V(G)$ with the property that any cycle of G contains at least one element of K. (Note that the vertex set of any maximal system of vertex disjoint cycles in G obviously satisfies this condition. Moreover, by a result of Erdős and Pósa, K can always be chosen so as to have fewer than $ck\log k$ elements.) We will refer to K, as the *kernel* of G. The vertices of G outside the kernel are called *external*. The external vertices induce a forest in G, and they can be classified according to which elements of K they are connected to. We color two external points with the same *color*, if and only if their sets of neighbors in K are the same. Thus, we obtain a coloring function $\phi:(V(G)-K)\to\Gamma$, where Γ is the set of colors and $|\Gamma|=2^{|K|}$.

Let $\mathscr{P}=\{P_1,P_2,\dots,P_m\}$ be a set of at most $|K|$ vertex-disjoint paths in $G-K$, and let v_i and v_i' denote the endpoints of P_i. (We do not exclude the possibility that $v_i=v_i'$, i.e., P_i consists of a single vertex.) The *type of* P_i (with respect to K) is defined by the colors of its endpoints:

$$\text{type }(P_i)=\begin{cases}[\phi(v_i)] \text{ if } v_i=v_i'\ ,\\ [\phi(v_i),\phi(v_i')] \text{ if } v_i\ne v_i'\ ;\end{cases}$$
$$\text{Type }(\mathscr{P})=[\text{type }(P_i)\,:\,1\le i\le m].$$

(We write $[\cdot]$ instead of $\{\cdot\}$ to indicate that some of the elements may be repeated, i.e., they form a *multiset*.) Furthermore, let $\mathscr{T}(G)$ be defined as the set of all Type($\mathscr{P}$), where $\mathscr{P}$ is a system of at most $|K|$ vertex-disjoint paths in G–K. Clearly, $\mathscr{T}(G)$ is closed under containment, i.e., $T\in\mathscr{G}(G)$ implies that every submultiset $T'\subseteq T$ also belongs to $\mathscr{T}(G)$.

For each member $T\in\mathscr{T}(G)$, fix a system $\mathscr{P}^T$ of vertex-disjoint paths with Type $(\mathscr{P}^T)=T$. Put

$$L=\bigcup_{T\in\mathscr{T}(G)}V(\mathscr{P}^T).$$

Evidently, L is a finite set and $L\cap K=\emptyset$. Since $G-K$ is a forest, there are only finitely many external vertices lying on some path connecting two elements of L. Let $\overline{L}$ denote the set obtained from L by adding all of these vertices.

Now $G-(K\cup\overline{L})$ falls into countably many connected components (trees) G_i $(i=1,2,\dots)$. Every G_i has at most one point adjacent to some element of $\overline{L}$. If such a point exists, then it is called the *root of* G_i. By the definition of $\overline{L}$, the root of G_i has only one neighbor in $\overline{L}$.

Next we describe a procedure that will enable us to add new vertices to G, without creating k vertex-disjoint cycles.

Let G_1 be a fixed component of $G-(K\cup\overline{L})$, and pick a color $\gamma\in\Gamma$ which occurs among the vertices of G_1 at least twice. Thus, one can find two distinct points $u_1, u_2 \in V(G_1)$ which are adjacent to the same elements of K. Let G' denote the graph obtained from G by adding a new vertex u of color γ and connecting it to any point $w \in V(G_1)$. That is,

$$V(G') = V(G) \cup \{u\},$$
$$E(G') = E(G) \cup \{uv : v \in K, u_1 v \in E(G)\} \cup \{uw\}.$$

Obviously, any cycle of G' passes through at least one element of K. In other words, K is also a kernel of G', hence it can be used to define $\mathcal{T}(G')$.

Lemma 1. $\mathcal{T}(G') = \mathcal{T}(G)$.

We have to prove only that $\mathcal{T}(G') \subseteq \mathcal{T}(G)$. Assume, in order to obtain a contradiction, that there is a system $\mathcal{P} = \{P_i : 1 \leq i \leq m\}$ of at most $|K|$ vertex-disjoint paths in $G' - K$ such that

$$T = \text{Type } (\mathcal{P}) = [\text{type } (P_i) : 1 \leq i \leq m] \in \mathcal{T}(G') - \mathcal{T}(G).$$

Suppose, without loss of generality, that $\mathcal{P}$ is a *minimal* system satisfying this condition, i.e.,

$$T_j = [\text{type } (P_i) : 1 \leq i \leq m, \quad i \neq j] \in \mathcal{T}(G)$$

for every j $(1 \leq j \leq m)$.

Clearly, one of the paths P_i (say, P_1) must contain the new vertex u, otherwise $T = \text{Type } (\mathcal{P}) \in \mathcal{T}(G)$. Moreover, u must be an endpoint of P_1, because the degree of u in $G' - K$ is 1. Let v_i and v_i' denote the (not necessarily distinct) endpoints of P_i. Thus, we can assume that $u = v_1$.

Let $G_1' \subseteq G'$ denote the tree obtained from G_1 by adding the vertex u and the edge uw.

Observe that no path P_j can be entirely contained in G_1'. To see this, recall that there is a system $\mathcal{P}^{T_j}$ of vertex-disjoint paths in L with Type $(\mathcal{P}^{T_j}) = T_j$. So, if P_j were in G_1' for some $j \neq 1$, then $\mathcal{P}^{T_j} \cup \{P_j\}$ would form a system of vertex-disjoint paths in G-K, whose type is T. If $P_1 \subseteq G_1'$, then consider the uniquely determined paths P_{11} and $P_{12} \subseteq G_1$ connecting v_1' to u_1 and u_2, respectively. At least one of them (say, P_{11}) is of the same type as P_1. Hence, $\mathcal{P}^{T_1} \cup \{P_{11}\}$ is a system of vertex-disjoint paths in G-K, whose type is T. In both cases we can conclude that $T \in \mathcal{T}(G)$, contradiction.

Thus, we can assume that v_1' is not in G_1'. This implies that G_1 has a root r, and P_1 must pass through r. Let P_{11} denote the (unique) path connecting v_1' and u_1 in G-K. Clearly, P_{11} also passes through r and type $(P_{11}) =$ type (P_1). Notice that P_{11} is disjoint from any P_j $(2 \leq j \leq m)$, otherwise P_j would lie entirely in G_1, contradicting our previous observation. Hence, $\{P_{11}, P_2, P_3, \ldots, P_m\}$ is a system of vertex-disjoint paths in G-K, whose type is T, which is again a contradiction. This completes the proof of Lemma 1.

Lemma 2. *G' has no k vertex-disjoint cycles.*

Assume, for contradiction, that there is a system $\{C_i : 1 \leq i \leq k\}$ of k vertex-disjoint cycles in G'. Since every cycle must visit K, the pieces of the C_i lying outside K form a system $\mathscr{P}'$ of at most $|K|$ vertex-disjoint pahts in G'-K. By Lemma 1, there exists a system $\mathscr{P}$ of vertex-disjoint paths in G-K such that Type $(\mathscr{P})$ = Type $(\mathscr{P}')$. For every cycle C_i, replace each piece lying outside K by the corresponding path in $\mathscr{P}$. Thus, we obtain k vertex-disjoint cycles in G, the desired contradiction establishing Lemma 2.

By the repeated application of the above procedure, we can add countably many new vertices to G, to obtain a graph G^* satisfying the conditions summarized in the following statement.

Lemma 3. *Let G be a countable graph without k vertex-disjoint cycles, and let $K, \overline{L} \subseteq V(G)$ be finite sets, as defined above.*

Then there exists a countable graph G^* with the following properties.

(i) G^* contains G as an induced subgraph;

(ii) G^* has no k vertex-disjoint cycles;

(iii) every cycle of G^* meets K.

Furthermore, let $\phi^*: (V(G^*) - K) \to \Gamma$ be a coloring assigning the same color to two vertices if and only if they are connected to the same elements of K. Let G_i^* $(i=1,2,\ldots)$ denote the connected components of $G^* - (K \cup \overline{L})$.

(iv) Each component G_i^* is connected to $\overline{L}$ by at most one edge;

(v) if $\gamma \in \Gamma$ is any color assigned to at least two points of G_i^*, then every vertex of G_i^* has infinitely many neighbors of color γ $(i=1,2,\ldots)$.

Let $\mathscr{G}_k^*$ be the family of all countable graphs that can be obtained as G^* for some $G \subset \mathscr{G}_k$. Obviously, $\mathscr{G}_k^* \subseteq \mathscr{G}_k$ and every element of $\mathscr{G}_k$ can be embedded into some element of $\mathscr{G}_k^*$ as an induced subgraph. On the other hand, $\mathscr{G}_k^*$ is clearly a *countable* family of graphs. To see this, we have to note only that

(a) there are only countably many different graphs that can be obtained as the restriction of some $G^* \in \mathscr{G}_k^*$ to the corresponding subset $K \cup \overline{L}$ (because K and $\overline{L}$ are finite);

(b) there are only countably many different colored graphs (trees) that can be obtained as G_i^* for some $G^* \in \mathscr{G}_k^*$ (because G_i^* is either finite or it is a tree whose every vertex has degree ω, and those points whose color does not appear anywhere else in G_i^* can be situated in this tree in countably many different ways);

(c) given K and $\overline{L}$, there are only countably many different ways that a colored tree G_i^* can be connected to these sets (because of Lemma 3 (iv)).

Hence, $c(\mathscr{G}_k) \leq \omega$.

Next we show that $c(\mathscr{G}_k) \geq \omega$. Let K_4 denote the complete graph on four vertices, and let G_0 be the graph obtained from the union of $k-1$ vertex-disjoint copies of K_4 by adding a vertex connected to every other point. Let $\mathscr{G}_0$ be the family of all *subdivisions* of G_0, i.e., the set of all graphs arising from G_0 by replacing its edges with independent paths. Clearly, $\mathscr{G}_0$ is a countable subfamily of $\mathscr{G}_k$. On the other hand, it is easy to check that, if G is a graph containing two subgraphs isomorphic to distinct elements of $\mathscr{G}_0$, then $G \notin \mathscr{G}_k$. Thus, $c(\mathscr{G}_k) \geq |\mathscr{G}_0| = \omega$, completing the proof.

The analogous result for countable graphs containing no k edge-disjoint cycles can be established by a similar argument.

Theorem 2. *Let $1<k<\omega$, and $\mathscr{G}'_k$ be the class of all countable graphs containing no k edge-disjoint cycles. Then $c(\mathscr{G}'_k)=\omega$.*

To see that $c(\mathscr{G}_k) \geq \omega$, we can repeat the above argument with the only difference that now G_0 has to be defined as the graph obtained from the union of $k-1$ vertex-disjoint *triangles* by adding a vertex connected to every other point. The minor modifications in the other part of the proof are left to the reader.

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